

THE DUALITY OF CONFORMALLY FLAT MANIFOLDS

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ABSTRACT. In a joint work with Saji, the second and the third authors gave an intrinsic formulation of wave fronts and proved a realization theorem for wave fronts in space forms. As an application, we show that the following four objects are essentially the same:

- conformally flat n -manifolds ($n \geq 3$) with admissible singular points (i.e. admissible GCF-manifolds),
- frontals as hypersurfaces in the lightcone Q_+^{n+1} ,
- frontals as hypersurfaces in the hyperbolic space H^{n+1} ,
- spacelike frontals as hypersurfaces in the de Sitter space S_1^{n+1} .

Recently, the duality of conformally flat Riemannian manifolds was discovered by several geometers. In our setting, this duality can be explained via the existence of a two-fold map of the congruence classes of admissible GCF-manifolds into that of frontals in H^{n+1} . It should be remarked that the dual conformally flat metric may have degenerate points even when the original conformally flat metric is positive definite. This is the reason why we consider conformally flat manifolds with singular points. In fact, the duality is an involution on the set of admissible GCF-manifolds. The case $n = 2$ requires a special treatment, since any Riemannian 2-manifold is conformally flat. At the end of this paper, we also determine the moduli space of isometric immersions of a given simply connected Riemannian 2-manifold into the lightcone Q_+^3 .

1. INTRODUCTION

We denote by Q_+^{n+1} the upper lightcone in Lorentz-Minkowski space \mathbf{L}^{n+2} . Izumiya [7] pointed out that the mean curvature H of a surface in Q_+^3 is equal to $-1/2$ times of the Gaussian curvature K , as Theorema Egregium of Gauss for surface theory in Q_+^3 . (It should be remarked that our notation is different from that in [7]. If one use Izumiya's notation, H coincides with K . This formula was found independently by the first author in [13, (2.7)] and in more general form in [15, (2.13)].) When $n \geq 3$, this corresponds to the fact that a conformally flat n -manifold can be isometrically immersed as a hypersurface in Q_+^{n+1} , and the second fundamental form is just its Schouten tensor (see Brinkmann [2] and Asperti-Dajczek [1]).

Recently, Izumiya [7], Espinar-Gálvez-Mira [4] and Liu-Jung [15] independently found a duality of hypersurfaces in Q_+^{n+1} . More precisely, Izumiya [7] explained this duality on hypersurfaces in Q_+^{n+1} as a bi-Legendrian fibration in contact geometry. The two distinct explicit formulas of the dual in the lightcone are given in [15] and [7, p. 332] respectively. On the other hand, Espinar, Gálvez and Mira [4] found the

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duality on conformally flat manifolds from the viewpoint of hypersurface theory in the hyperbolic space, and found that the inverses of the eigenvalues of their Schouten tensors coincide with the eigenvalues of the dual Schouten tensors. It should be remarked that the dual metric of a conformally flat Riemannian metric might degenerate, in general.

On the other hand, in a joint work [21] with Saji, the second and the third authors gave the definition of a frontal bundle, and proved a realization of it as a wave front in space forms, which is a generalization of the fundamental theorem of surface theory. In this paper, as an application of this, we define ‘admissible generalized conformally flat manifolds’ (or ‘admissible GCF-manifolds’) as a class of conformally flat manifolds with admissible singular points, and show that the above duality operation is an involution on this class. Also, we give an explicit formula for dual metrics, and remark that the Schouten tensors are invariant under the duality operation. Moreover, as a refinement of the result in [4], under the assumption that M^n ($n \geq 3$) is 1-connected (i.e. connected and simply connected), we show that this duality comes from the existence of the two-fold map

$$\Psi : \mathcal{GCF}(M^n) \longrightarrow \mathcal{M}_{Fr}(M^n, H^{n+1})$$

of the moduli space $\mathcal{GCF}(M^n)$ of admissible GCF-manifolds into the moduli space $\mathcal{M}_{Fr}(M^n, H^{n+1})$ of frontals in hyperbolic space H^{n+1} . To prove the existence of the map Ψ , we apply the realization theorem of intrinsic wave fronts given in [21].

Finally, we consider the 2-dimensional case, and determine the moduli space of isometric immersions of a given simply connected Riemannian 2-manifold into Q_+^3 .

2. THE DUALITY OF CONFORMALLY FLAT MANIFOLDS

A Riemannian n -manifold (M^n, g) is called *conformally flat* if for each point $p \in M^n$, there exists a neighborhood $U(\subset M^n)$ of p and a C^∞ -function σ on U such that $e^{2\sigma}g$ is a metric with vanishing sectional curvature. When $n \geq 4$, (M^n, g) is conformally flat if and only if the conformal curvature tensor

$$(2.1) \quad W_{ijkl} := R_{ijkl} + A_{ik}g_{jl} - A_{il}g_{jk} + A_{jl}g_{ik} - A_{jk}g_{il}$$

vanishes identically on M^n , where (u^1, \dots, u^n) is a local coordinate system of M^n ,

$$(2.2) \quad A := \frac{1}{n-2} \sum_{i,j} \left(R_{ij} - \frac{S_g}{2(n-1)} g_{ij} \right) du^i \otimes du^j$$

is called the *Schouten tensor*, g_{ij} , R_{ijkl} , R_{ij} are the components of the metric g , the curvature tensor of g , and the Ricci tensor of g respectively, and S_g denotes the scalar curvature. When $n = 3$, (M^3, g) is conformally flat if and only if A in (2.2) is a Codazzi tensor, that is, ∇A is a symmetric 3-tensor, where ∇ is the Levi-Civita connection of (M^3, g) . (When $n \geq 4$, conformal flatness implies that A is a Codazzi tensor because of the second Bianchi identity.) When $n = 2$, all Riemannian metrics are conformally flat.

To formulate conformally flat manifolds with singularities, we need to define the following:

Definition ([18], [19], [21]). Let \mathcal{E} be a vector bundle over an n -manifold M^n ($n \geq 1$) of rank n , and $\varphi : TM^n \rightarrow \mathcal{E}$ a bundle homomorphism, where TM^n is the tangent bundle of M^n . Suppose that \mathcal{E} has a metric $\langle \cdot, \cdot \rangle$ and a metric connection

D . Then $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ is called a *coherent tangent bundle* if φ satisfies the condition

$$(2.3) \quad D_X \varphi(Y) - D_Y \varphi(X) - \varphi([X, Y]) = 0$$

for any C^∞ -vector fields X, Y on M^n . Let $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ be a coherent tangent bundle. Then $p \in M^n$ is called a *singular point* of φ if the linear map $\varphi_p : T_p M^n \rightarrow \mathcal{E}_p$ is not injective, where \mathcal{E}_p is the fiber of \mathcal{E} at p . On the other hand, $p \in M^n$ is called a *regular point* if it is not a singular point. We denote by \mathcal{R}_{M^n} or $\mathcal{R}_{M^n, \varphi}$ the set of regular points of φ .

Coherent tangent bundles can be considered as a generalization of Riemannian metrics. In fact, the pull-back metric $g := \varphi^* \langle \cdot, \cdot \rangle$ gives a Riemannian metric on \mathcal{R}_{M^n} and the Levi-Civita connection ∇ of g coincides with the pull-back of the connection D by φ because of the condition (2.3). Moreover, one can prove the Gauss-Bonnet formula when $n = 2$ (see [18], [19] and [21]).

Example 2.1. Let M^n and N^{n+1} be C^∞ -manifolds of dimension n and of dimension $n + 1$, respectively. The projectified cotangent bundle $P(T^*N^{n+1})$ has a canonical contact structure. A C^∞ -map $f : M^n \rightarrow N^{n+1}$ is called a *frontal* if f lifts to a Legendrian map L_f , i.e., a C^∞ -map $L_f : M^n \rightarrow P(T^*N^{n+1})$ such that the image $dL_f(TM^n)$ of the tangent bundle TM^n lies in the contact hyperplane field on $P(T^*N^{n+1})$. Moreover, f is called a *wave front* or a *front* if it lifts to a Legendrian immersion L_f . Frontals (and therefore fronts) generalize immersions, as they allow for singular points. A frontal f is said to be co-orientable if its Legendrian lift L_f can lift up to a C^∞ -map into the cotangent bundle T^*N^{n+1} .

Now, we fix a Riemannian metric \tilde{g} on N^{n+1} . Then, it can be easily checked that a C^∞ -map

$$f : M^n \longrightarrow N^{n+1}$$

is a frontal if and only if for each $p \in M^n$, there exists a neighborhood U of p and a unit C^∞ -vector field ν of N^{n+1} along f defined on U such that $\tilde{g}(df(X), \nu) = 0$ holds for any vector fields X on U (that is, ν is a locally defined unit normal vector field). Moreover, if the locally defined unit normal vector field $\nu : U \rightarrow T_1 N^{n+1}$ can be taken to be an immersion for each $p \in M^n$, f is called a *front*, where $T_1 N^{n+1}$ is the unit tangent bundle of (N^{n+1}, \tilde{g}) . We denote by \mathcal{E}_f the subbundle of the pull-back bundle $f^*(TN^{n+1})$ consisting of vectors perpendicular to ν . Then

$$\varphi_f : TM^n \ni X \longmapsto df(X) \in \mathcal{E}_f$$

gives a bundle homomorphism. Let $\tilde{\nabla}$ be the Levi-Civita connection on N^{n+1} . Then by taking the tangential part of $\tilde{\nabla}$, a connection D on \mathcal{E}_f satisfying (2.3) is induced. Let $\langle \cdot, \cdot \rangle$ be a metric on \mathcal{E}_f induced from the Riemannian metric on N^{n+1} , then D is a metric connection on \mathcal{E}_f . Thus we get a coherent tangent bundle $(M^n, \mathcal{E}_f, \langle \cdot, \cdot \rangle, D, \varphi_f)$.

In this setting, we define the following

Definition. A given coherent tangent bundle $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ ($n \geq 3$) is called a *generalized conformally flat manifold* or a *GCF-manifold* if the regular set \mathcal{R}_{M^n} of φ is dense in M^n and the pull-back metric $g := \varphi^* \langle \cdot, \cdot \rangle$ is conformally flat on \mathcal{R}_{M^n} .

Example 2.2. Let (N^n, \tilde{g}) ($n \geq 3$) be a conformally flat Riemannian n -manifold, and $f : M^n \rightarrow N^n$ a C^∞ -map whose regular set is dense in M^n . Let $\mathcal{E}_f := f^*(TN^n)$ be the pull-back bundle of TN^n by f . Then \tilde{g} induces a positive definite metric $\langle \cdot, \cdot \rangle$ on \mathcal{E}_f , and the pull-back D of the Levi-Civita connection of \tilde{g} gives a metric connection on $(\mathcal{E}_f, \langle \cdot, \cdot \rangle)$. We set

$$\varphi_f := df : TM^n \longrightarrow \mathcal{E}_f.$$

Then we have a coherent tangent bundle $(M^n, \mathcal{E}_f, \langle \cdot, \cdot \rangle, D, \varphi_f)$ which is a GCF-manifold because (N^n, \tilde{g}) is conformally flat.

Definition. Let $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ ($n \geq 3$) be a GCF-manifold. A point $p \in \mathcal{R}_{M^n}$ is called a *parabolic point* if the Schouten tensor A of the induced metric $g = \varphi^* \langle \cdot, \cdot \rangle$ degenerates. We denote by $\mathcal{R}_{M^n}^* (\subset \mathcal{R}_{M^n})$ the set of non-parabolic points.

The following assertion is the explicit description of the duality of conformally flat manifolds:

Theorem 2.3. *Let $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ ($n \geq 3$) be a GCF-manifold, and $g = \varphi^* \langle \cdot, \cdot \rangle$ the induced metric. Then*

$$(2.4) \quad \check{g} := \sum_{i,j,a,b} A_{ia} A_{jb} g^{ab} du^i \otimes du^j$$

gives a conformally flat metric on $\mathcal{R}_{M^n}^$ (called the dual metric), where (u^1, \dots, u^n) is a local coordinate system, (g^{ab}) is the inverse matrix of (g_{ij}) and A is the Schouten tensor of g . Moreover, the Schouten tensor of \check{g} coincides with A .*

One can prove the assertion by a direct calculation. We give an alternative proof in Section 4. The following assertion follows immediately, which was proved in [4]:

Corollary 2.4 ([4]). *The eigenvalues of the Schouten tensor with respect to \check{g} are the inverses of those of A with respect to g .*

As seen in Example 2.2, the class of GCF-manifolds might be too wide. We now define a subclass of GCF-manifolds which admits the duality and also the conformal changes of metrics as follows: Set

$$(2.5) \quad \hat{A} := \sum_{i,j,a} g^{ia} A_{aj} \frac{\partial}{\partial u^i} \otimes du^j,$$

which is a tensor defined on \mathcal{R}_{M^n} , where $g = \varphi^* \langle \cdot, \cdot \rangle$ is the induced metric and A is the Schouten tensor of g on \mathcal{R}_{M^n} .

Definition. Let $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ ($n \geq 3$) be a GCF-manifold. Then it is called *admissible* if $\mathcal{R}_{M^n}^*$ is dense in M^n and the tensor \hat{A} induces a new bundle homomorphism

$$(2.6) \quad \check{\varphi} : TM^n \ni v \longmapsto \varphi \circ \hat{A}(v) \in \mathcal{E},$$

namely, the homomorphism $TM^n|_{\mathcal{R}_{M^n}} \ni v \mapsto \varphi \circ \hat{A}(v) \in \mathcal{E}$ can be smoothly extended to the whole of TM^n .

In this setting, we can formulate the duality of conformally flat manifolds as follows:

Theorem 2.5. *Let $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ ($n \geq 3$) be an admissible GCF-manifold. Then by replacing φ by $\check{\varphi}$ in (2.6), $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \check{\varphi})$ is also an admissible GCF-manifold. Moreover $\check{\check{\varphi}}$ coincides with φ .*

We prove this assertion in Section 4.

Remark 2.6. We can prove Theorems 2.3 and 2.5 under the assumption that $\langle \cdot, \cdot \rangle$ is a non-degenerate symmetric bilinear form. So, for example, the duality also holds for Lorentzian conformally flat manifolds.

Moreover, we also prove the following in Section 3:

Proposition 2.7. *Let $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ ($n \geq 3$) be an admissible GCF-manifold. Then for a given C^∞ -function σ on M^n , there exist a connection D^σ and a bundle homomorphism $\varphi_\sigma : TM^n \rightarrow \mathcal{E}$ such that $(M^n, \mathcal{E}, e^{2\sigma} \langle \cdot, \cdot \rangle, D^\sigma, \varphi_\sigma)$ is a GCF-manifold.*

The conformal change of the metric g of a GCF-manifold as in Proposition 2.7 is canonical in the sense that it is induced from an extrinsic conformal change of the metric in the lightcone. If a GCF-manifold has no singular points, this coincides with the usual conformal change of the conformally flat metric. However, if a GCF-manifold admits singular points, our conformal change may not preserve the admissibility in general, since the dual metric \tilde{g} may diverge at a degenerate point of g (see Remark 4.7). We also remark that the singular sets may not be stable under conformal changes.

3. FRONTAL BUNDLES

Let M^n be an oriented n -manifold ($n \geq 1$) and $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ a coherent tangent bundle over M^n . Let $\psi : TM^n \rightarrow \mathcal{E}$ be another bundle homomorphism satisfying the following conditions

- (1) $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \psi)$ is also a coherent tangent bundle,
- (2) the pair (φ, ψ) of bundle homomorphisms satisfies a compatibility condition

$$(3.1) \quad \langle \varphi(X), \psi(Y) \rangle = \langle \varphi(Y), \psi(X) \rangle,$$

where $X, Y \in T_p M^n$ and $p \in M^n$.

Then $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi, \psi)$ is called a *frontal bundle* (see [21]). The bundle homomorphisms φ and ψ are called the *first homomorphism* and the *second homomorphism*, respectively. We set

$$\begin{aligned} I(X, Y) &:= \langle \varphi(X), \varphi(Y) \rangle, \\ II(X, Y) &:= -\langle \varphi(X), \psi(Y) \rangle, \\ III(X, Y) &:= \langle \psi(X), \psi(Y) \rangle \end{aligned}$$

for $X, Y \in T_p M^n$ ($p \in M^n$), and call them *the first, the second and the third fundamental forms*, respectively. They are all symmetric covariant tensors on M^n . A frontal bundle $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi, \psi)$ is called a *front bundle* if

$$(3.2) \quad \text{Ker}(\varphi_p) \cap \text{Ker}(\psi_p) = \{0\}$$

holds for each $p \in M^n$. The conditions for φ and ψ in the definition of frontal bundles are symmetric in φ and ψ , so we can exchange their roles. (Then the first fundamental form becomes the third fundamental form.)

Example 3.1. Let $(\tilde{N}^{n+1}(c), \tilde{g})$ be the $(n+1)$ -dimensional 1-connected space form of constant curvature c , and denote by $\tilde{\nabla}$ the Levi-Civita connection of $\tilde{N}^{n+1}(c)$. Let $f : M^n \rightarrow \tilde{N}^{n+1}(c)$ be a co-orientable frontal, that is, the unit normal vector field ν is defined globally on M^n . Since the coherent tangent bundle \mathcal{E}_f given in Example 2.1 is orthogonal to ν , we can define a bundle homomorphism

$$\psi_f : T_p M^n \ni X \mapsto \tilde{\nabla}_X \nu \in (\mathcal{E}_f)_p \quad (p \in M^n),$$

which can be considered as the shape operator of f . Then $(M^n, \mathcal{E}_f, \langle \cdot, \cdot \rangle, D, \varphi_f, \psi_f)$ is a frontal bundle (see Fact 3.2 later). Moreover, this is a front bundle if and only if f is a front, which is equivalent to $I + III$ being positive definite.

Fact 3.2 ([21]). *Let $f : M^n \rightarrow \tilde{N}^{n+1}(c)$ be a co-orientable frontal, and ν a unit normal vector field. Then $(M^n, \mathcal{E}_f, \langle \cdot, \cdot \rangle, D, \varphi_f, \psi_f)$ as in Example 3.1 is a frontal bundle. Moreover, the following identity (i.e. the Gauss equation) holds:*

$$(3.3) \quad \langle R^D(X, Y)v, w \rangle = c \det \begin{pmatrix} \langle \varphi(Y), v \rangle & \langle \varphi(Y), w \rangle \\ \langle \varphi(X), v \rangle & \langle \varphi(X), w \rangle \end{pmatrix} \\ + \det \begin{pmatrix} \langle \psi(Y), v \rangle & \langle \psi(Y), w \rangle \\ \langle \psi(X), v \rangle & \langle \psi(X), w \rangle \end{pmatrix},$$

where $\varphi = \varphi_f$ and $\psi = \psi_f$, X and Y are vector fields on M^n , v and w are sections of \mathcal{E}_f , and R^D is the curvature tensor of the connection D :

$$R^D(X, Y)v := D_X D_Y v - D_Y D_X v - D_{[X, Y]}v.$$

Furthermore, this frontal bundle is a front bundle if and only if f is a front.

Two frontal bundles over M^n are *isomorphic* or *equivalent* if there exists an orientation preserving bundle isomorphism between them which preserves the inner products, the connections and the bundle maps.

Fact 3.3 ([21]). *Let $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi, \psi)$ be a frontal bundle over a 1-connected manifold M^n ($n \geq 1$) satisfying (3.3), where c is a real number. Then there exists a frontal $f : M^n \rightarrow \tilde{N}^{n+1}(c)$ such that \mathcal{E} is isomorphic to \mathcal{E}_f induced from f as in Fact 3.2. Moreover, such an f is unique up to orientation preserving isometries of $\tilde{N}^{n+1}(c)$.*

Recall that Q_+^{n+1} (resp. Q_-^{n+1}) is the upper (resp. lower) lightcone in Lorentz-Minkowski space $(\mathbf{L}^{n+2}, \langle \cdot, \cdot \rangle)$ of signature $(- + \cdots +)$:

$$Q_{\pm}^{n+1} = \{z = (z^0, z^1, \dots, z^{n+1}) \in \mathbf{L}^{n+2}; \langle z, z \rangle = 0, \pm z^0 > 0\}.$$

From now on, we shall apply Fact 3.3 to hypersurface theory in the lightcone Q_+^{n+1} : A C^∞ -map $x : M^n \rightarrow Q_+^{n+1}$ is called a *spacelike frontal* if there exists another C^∞ -map $y : M^n \rightarrow Q_-^{n+1}$ such that

$$(3.4) \quad \langle x, x \rangle = \langle y, y \rangle = 0, \quad \langle x, y \rangle = 1,$$

$$(3.5) \quad \langle dx, y \rangle = \langle dy, x \rangle = 0,$$

where $\langle dx, y \rangle$ and $\langle dy, x \rangle$ are considered as 1-forms, for example, $\langle dx, y \rangle$ is defined by $TM^n \ni X \mapsto \langle dx(X), y \rangle \in \mathbf{R}$.

In this setting, y is called the *dual* of x . Then y is also a frontal. Moreover, if the pair

$$(x, y) : M^n \longrightarrow Q_+^{n+1} \times Q_-^{n+1}$$

gives an immersion, x is called a *spacelike front*.

Remark 3.4. Let $x : M^n \rightarrow Q_+^{n+1}$ be a spacelike frontal as above. Then the linear map

$$L_p : T_{x(p)}Q_+^{n+1} \ni v \mapsto \langle y(p), v \rangle \in \mathbf{R}$$

induces a Legendrian lift

$$[L] : M^n \ni p \mapsto [L_p] \in P(T^*Q_+^{n+1})$$

of x , where $T^*Q_+^{n+1} \ni \alpha \mapsto [\alpha] \in P(T^*Q_+^{n+1})$ is the canonical projection. Thus, a spacelike frontal is a frontal as in Example 2.1. Moreover, since

$$\{x(p), y(p)\}^\perp = \{x(p) - y(p), x(p) + y(p)\}^\perp$$

and $x(p) - y(p)$ is a timelike vector, the kernel of L_p is a spacelike vector space for each $p \in M^n$.

Conversely, a frontal in Q_+^{n+1} is spacelike if and only if it has a Legendrian lift $[L] : M^n \ni p \mapsto [L_p] \in P(T^*Q_+^{n+1})$, such that the kernel of L_p is a spacelike vector space for each $p \in M^n$. In fact, if the kernel $Z_p (\subset T_{x(p)}Q_+^{n+1})$ of L_p is a spacelike vector space, then the orthogonal complement $(Z_p)^\perp$ in \mathbf{L}^{n+2} is a Lorentzian plane containing $x(p)$. Thus, there exists a unique null vector $y(p) \in (Z_p)^\perp$ such that $\langle x(p), y(p) \rangle = 1$, which induces a C^∞ -map $y : M^n \rightarrow Q_-^{n+1}$ and x is a spacelike frontal.

Remark 3.5. There is a spacelike frontal which is not a front. In fact, we set

$$M^n := S^n \setminus \{(0, \dots, 0, \pm 1)\},$$

where

$$S^n = \{(u^1, \dots, u^{n+1}) \in \mathbf{R}^{n+1} ; (u^1)^2 + \dots + (u^{n+1})^2 = 1\},$$

and set

$$x : M^n \ni (u^1, \dots, u^{n+1}) \mapsto \frac{1}{\sqrt{2}} \left(1, \frac{u^1}{\sqrt{1 - (u^{n+1})^2}}, \dots, \frac{u^n}{\sqrt{1 - (u^{n+1})^2}}, 0 \right) \in Q_+^{n+1}.$$

Then

$$y : M^n \ni (u^1, \dots, u^{n+1}) \mapsto \frac{1}{\sqrt{2}} \left(-1, \frac{u^1}{\sqrt{1 - (u^{n+1})^2}}, \dots, \frac{u^n}{\sqrt{1 - (u^{n+1})^2}}, 0 \right) \in Q_-^{n+1}$$

gives the dual of x . Thus x is a frontal, but not front, since the image of (x, y) lies on an $(n-1)$ -dimensional submanifold of $Q_+^{n+1} \times Q_-^{n+1}$. On the other hand, there is a spacelike front which is not an immersion (see Corollary 4.6).

We consider two canonical projections

$$\Pi_\pm : Q_\pm^{n+1} \longrightarrow S_\pm^n := \{z^0 = \pm 1\} \cap Q_\pm^{n+1},$$

where $(z^0, z^1, \dots, z^{n+1})$ is the canonical coordinate system of \mathbf{L}^{n+2} , and S_+^n (resp. S_-^n) is the sphere embedded in Q_+^{n+1} (resp. in Q_-^{n+1}). We set

$$(3.6) \quad G_+ := \Pi_+ \circ x, \quad G_- := \Pi_- \circ y,$$

which are called the *Gauss maps* of x and y , respectively. In this setting, the following assertion can be proved immediately:

Lemma 3.6. *Let $x : M^n \rightarrow Q_+^{n+1}$ ($n \geq 1$) be a spacelike frontal. Then $\langle dx, dx \rangle$ is non-degenerate if and only if G_+ is an immersion. In particular, x itself is an immersion.*

Remark 3.7. There is an immersed hypersurface in Q_+^{n+1} , which is not spacelike. For example, the sub-lightcone

$$\{(z^0, z^1, \dots, z^{n+1}) \in Q_+^{n+1}; z^{n+1} = 0\}$$

is an immersed hypersurface but not spacelike. Its Gauss map G_+ degenerates everywhere on it.

Let $x : M^n \rightarrow Q_+^{n+1}$ ($n \geq 1$) be a spacelike frontal. Then

$$f := \frac{1}{\sqrt{2}}(x - y) : M^n \longrightarrow H^{n+1} = \tilde{N}^{n+1}(-1) \subset \mathbf{L}^{n+2}$$

gives a frontal with the unit normal vector field

$$\nu := \frac{1}{\sqrt{2}}(x + y) : M^n \longrightarrow S_1^{n+1} \subset \mathbf{L}^{n+2},$$

where H^{n+1} is the hyperbolic $(n+1)$ -space and S_1^{n+1} is the de Sitter $(n+1)$ -space (i.e. the simply connected complete Lorentzian space form of constant sectional curvature 1):

$$(3.7) \quad \begin{aligned} H^{n+1} &= \{z = (z^0, z^1, \dots, z^{n+1}) \in \mathbf{L}^{n+2}; \langle z, z \rangle = -1, z^0 > 0\}, \\ S_1^{n+1} &= \{z = (z^0, z^1, \dots, z^{n+1}) \in \mathbf{L}^{n+2}; \langle z, z \rangle = 1\}. \end{aligned}$$

Here, G_+ and G_- as in (3.6) are called the *hyperbolic Gauss maps* of f , see [7], [3] and [4]. We set

$$\xi := dx = \frac{1}{\sqrt{2}}(df + d\nu), \quad \zeta := dy = -\frac{1}{\sqrt{2}}(df - d\nu).$$

Since x is a spacelike frontal, we get a frontal bundle $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \xi, \zeta)$, where D is a metric connection of \mathcal{E} induced by the canonical connection in \mathbf{L}^{n+2} by taking the tangential components. Moreover, the following assertion holds:

Theorem 3.8. *Let M^n ($n \geq 1$) be a 1-connected n -dimensional manifold and $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \xi, \zeta)$ a frontal bundle satisfying*

$$(3.8) \quad \begin{aligned} \langle R^D(X, Y)v, w \rangle &= \det \begin{pmatrix} \langle \xi(Y), v \rangle & \langle \xi(Y), w \rangle \\ \langle \zeta(X), v \rangle & \langle \zeta(X), w \rangle \end{pmatrix} \\ &\quad + \det \begin{pmatrix} \langle \zeta(Y), v \rangle & \langle \zeta(Y), w \rangle \\ \langle \xi(X), v \rangle & \langle \xi(X), w \rangle \end{pmatrix}, \end{aligned}$$

where X and Y are vector fields on M^n , and v, w are sections of \mathcal{E} . Then there exists a spacelike frontal

$$x : M^n \longrightarrow Q_+^{n+1}$$

with its dual y such that $\langle dx, dx \rangle$, $-\langle dx, dy \rangle$ and $\langle dy, dy \rangle$ are the first, the second, and the third fundamental forms of $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \xi, \zeta)$, respectively. Conversely, any spacelike frontal of M^n into Q_+^{n+1} is given in this manner.

Proof. We set

$$\varphi := \frac{1}{\sqrt{2}}(\xi - \zeta), \quad \psi := \frac{1}{\sqrt{2}}(\xi + \zeta).$$

Then one can easily check that $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi, \psi)$ satisfies the conditions of Fact 3.3 for $c = -1$. In fact, (3.3) with $c = -1$ is equivalent to (3.8). Since M^n is simply connected, Fact 3.3 and a standard continuation argument imply that there

is a frontal $f : M^n \rightarrow H^{n+1}$ with unit normal vector field $\nu : M^n \rightarrow S_1^{n+1}$. We now set

$$x := \frac{1}{\sqrt{2}}(f + \nu), \quad y := -\frac{1}{\sqrt{2}}(f - \nu).$$

Then x (resp. y) is a map into Q_+^{n+1} (resp. Q_-^{n+1}) and it can be easily checked that $\langle dx, dx \rangle$, $-\langle dx, dy \rangle$ and $\langle dy, dy \rangle$ are equal to the first, the second, and the third fundamental forms of $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \xi, \zeta)$. Conversely, let x be a spacelike frontal in Q_+^{n+1} , then there exists a dual frontal y (cf. (3.4) and (3.5)) such that

$$f := \frac{1}{\sqrt{2}}(x - y), \quad \nu := \frac{1}{\sqrt{2}}(x + y).$$

Then $f : M^n \rightarrow H^{n+1}$ is a frontal, and ν is its unit normal vector field. As seen in Example 3.1, f induces a frontal bundle $(M^n, \mathcal{E}_f, \langle \cdot, \cdot \rangle, D, \varphi_f, \psi_f)$. If we set

$$\xi := \frac{1}{\sqrt{2}}(\varphi_f - \psi_f), \quad \zeta := -\frac{1}{\sqrt{2}}(\varphi_f + \psi_f),$$

then $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \xi, \zeta)$ is the desired frontal bundle satisfying (3.8). \square

4. CONFORMALLY FLAT MANIFOLDS AND HYPERSURFACES IN H^{n+1}

Theorem 2.3 follows immediately from the following

Lemma 4.1. *Let $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \xi, \zeta)$ be a frontal bundle over a 1-connected manifold M^n ($n \geq 3$) satisfying (3.8). Suppose that the regular set \mathcal{R}_{M^n} of ξ is dense in M^n . Then the first fundamental form $g := \xi^* \langle \cdot, \cdot \rangle$ is a conformally flat metric on \mathcal{R}_{M^n} , and the Schouten tensor A of g coincides with $-II$, where II is the second fundamental form of the frontal bundle. Moreover, the third fundamental form $\check{g} := \zeta^* \langle \cdot, \cdot \rangle$ satisfies (2.4).*

Proof. The curvature tensor R_g of g is related to R^D by

$$g(R_g(X, Y)Z, W) = \langle R^D(X, Y)\xi(Z), \xi(W) \rangle,$$

where X, Y, Z, W are vector fields on M^n . Substituting (3.8) to $v = \xi(Z)$ and $w = \xi(W)$, and by contraction, the Ricci tensor Ric_g is given by

$$(4.1) \quad \text{Ric}_g = -\text{Trace}_I(II)g - (n-2)II,$$

where Trace_I denotes the trace with respect to the first fundamental form $g = I = \xi^* \langle \cdot, \cdot \rangle$. Then the scalar curvature S_g is given by

$$(4.2) \quad S_g = -2(n-1)\text{Trace}_I(II).$$

When $n = 2$, this implies the equivalency between the Gaussian curvature and the mean curvature mentioned in the introduction. On the other hand, when $n \geq 3$, by (4.1) and (4.2), we have that

$$-II = \frac{1}{n-2} \left(\text{Ric}_g - \frac{S_g}{2(n-1)}g \right) = A,$$

where A is the Schouten tensor as in (2.2). Then A is a Codazzi tensor because of (2.3) for ξ and η . Moreover, if $n \geq 4$, one can easily see that the equation (3.8) with $v = \xi(Z)$ and $w = \xi(W)$ is equivalent to having that the conformal

curvature tensor as in (2.1) vanishes identically on \mathcal{R}_{M^n} . Let (u^1, \dots, u^n) be a local coordinate system of \mathcal{R}_{M^n} . We set

$$dy(\partial_i) = \sum_j \lambda_i^j dx(\partial_j) + c_1 x + c_2 y,$$

where $\partial_i = \partial/\partial u^i$. Since $\langle x, dy \rangle = \langle x, dx \rangle = 0$ and $\langle y, dx \rangle = \langle y, dy \rangle = 0$, we have that $c_1 = c_2 = 0$ and

$$A_{ij} = \langle dx(\partial_i), dy(\partial_j) \rangle = \sum_k \lambda_i^k g_{kj}.$$

Then it holds that $\lambda_i^j = \sum_k A_{ik} g^{kj}$ and

$$\check{g}_{ij} = \langle dy(\partial_i), dy(\partial_j) \rangle = \sum_{a,b} \lambda_i^a \lambda_j^b g_{ab} = \sum_{a,b} A_{ia} g^{ab} A_{bj},$$

which proves the assertion, where (g^{ab}) is the inverse matrix of (g_{ij}) . \square

Proof of Theorem 2.5. Let $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ be an admissible GCF-manifold. By (2.6), $\check{\varphi}$ gives a bundle homomorphism between TM^n and \mathcal{E} . By the previous Lemma 4.1, $\check{\varphi}^* \langle \cdot, \cdot \rangle$ is a conformally flat metric on $\mathcal{R}_{M^n}^*$. By Theorem 2.3, the Schouten tensor A is common in two metrics $\varphi^* \langle \cdot, \cdot \rangle$ and $\check{\varphi}^* \langle \cdot, \cdot \rangle$. As pointed out in Section 2, the second Bianchi identity with respect to $\varphi^* \langle \cdot, \cdot \rangle$ implies that A is a Codazzi tensor on \mathcal{R}_M^* , and then it is equivalent to the relation

$$(4.3) \quad \nabla_X \hat{A}(Y) - \nabla_Y \hat{A}(X) - \hat{A}([X, Y]) = 0,$$

where \hat{A} is given in (2.5) and ∇ is the Levi-Civita connection of $\varphi^* \langle \cdot, \cdot \rangle$. Since $(\mathcal{R}_{M^n}^*, TM^n|_{\mathcal{R}_{M^n}^*}, \varphi^* \langle \cdot, \cdot \rangle, \nabla, \hat{A})$ is isomorphic to $(\mathcal{R}_{M^n}^*, \mathcal{E}|_{\mathcal{R}_{M^n}^*}, \langle \cdot, \cdot \rangle, D, \check{\varphi})$ by φ , the identity (4.3) yields that

$$(4.4) \quad D_X \check{\varphi}(Y) - D_Y \check{\varphi}(X) - \check{\varphi}([X, Y]) = 0$$

holds on $\mathcal{R}_{M^n}^*$. Since $\mathcal{R}_{M^n}^*$ is dense in M^n , (4.4) holds on the whole of M^n . Thus $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \check{\varphi})$ is also a GCF-manifold. Since the set of regular points $\mathcal{R}_{M^n, \check{\varphi}}$ of $\check{\varphi}$ contains $\mathcal{R}_{M^n}^*$, $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \check{\varphi})$ is an admissible GCF-manifold. Then as seen in the proof of Lemma 4.1, $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi, \check{\varphi})$ is a frontal bundle over M^n satisfying (3.8) for $\xi = \varphi$ and $\zeta = \check{\varphi}$. Since the condition (3.8) is symmetric with respect to ξ and ζ , by switching the roles of φ and $\check{\varphi}$, we can conclude that $\check{\check{\varphi}}$ coincides with φ . \square

Let $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ be an admissible GCF-manifold. Then as seen in the above proof of Theorem 2.5, it induces a bundle homomorphism $\check{\varphi} : TM^n \rightarrow \mathcal{E}$ satisfying (3.8) for $\xi = \varphi$ and $\zeta = \check{\varphi}$. As seen in the proof of Theorem 3.8, if we set

$$\psi_1 := \frac{1}{\sqrt{2}}(\varphi - \check{\varphi}), \quad \psi_2 := \frac{1}{\sqrt{2}}(\varphi + \check{\varphi}),$$

then $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \psi_1, \psi_2)$ satisfies the condition of Fact 3.3 for $c = -1$. Then we get a frontal $f : \tilde{M}^n \rightarrow H^{n+1}$, where \tilde{M}^n is the universal covering of M^n . Here, f is determined up to rigid motions in H^{n+1} . Let ν be the unit normal vector field of f . Then it induces a spacelike wave front (cf. [21, Section 2]) $\nu : M^n \rightarrow S_1^{n+1}$, where S_1^{n+1} is the de Sitter space. The two frontals f and ν are mutually dual

objects. The realization theorem of spacelike fronts in S_1^{n+1} is proved in [21] using the duality. In particular, if M^n is 1-connected, we get two maps

$$\begin{aligned}\Psi : \mathcal{GCF}(M^n) &\longrightarrow \mathcal{M}_{Fr}(M^n, H^{n+1}), \\ \Psi^* : \mathcal{GCF}(M^n) &\longrightarrow \mathcal{M}_{Fr}(M^n, S_1^{n+1}),\end{aligned}$$

at the same time, where $\mathcal{GCF}(M^n)$ is the set of admissible GCF-manifolds modulo structure-preserving bundle isomorphisms and the set $\mathcal{M}_{Fr}(M^n, H^{n+1})$ (resp. $\mathcal{M}_{Fr}(M^n, S_1^{n+1})$) is the set of congruent classes of frontals in H^{n+1} (resp. spacelike frontals in S_1^{n+1}). The maps Ψ and Ψ^* are both two-fold maps, since Ψ (resp. Ψ^*) takes the same values for a dual GCF-manifold as for the given GCF-manifold.

We fix an admissible GCF-manifold $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$. By Lemma 3.6, the intersection of two regular sets of the hyperbolic Gauss maps $G_\pm : M^n \rightarrow S^n$ of f coincides with the set $\mathcal{R}_{M^n}^*$. Let $\mathcal{M}_{Fr}^*(M^n, H^{n+1})$ (resp. $\mathcal{M}_{Fr}^*(M^n, S_1^{n+1})$) the subset of $\mathcal{M}_{Fr}(M^n, H^{n+1})$ (resp. $\mathcal{M}_{Fr}(M^n, S_1^{n+1})$) consists of wave fronts (resp. spacelike wave fronts) whose pairs of Gauss maps (G_+, G_-) both have dense regular sets in M^n . Thus we get the following

Theorem 4.2. *Let M^n ($n \geq 3$) be a 1-connected manifold. Suppose that $\mathcal{GCF}(M^n)$ is non-empty. Then $\Psi : \mathcal{GCF}(M^n) \rightarrow \mathcal{M}_{Fr}^*(M^n, H^{n+1})$ (resp. $\Psi^* : \mathcal{GCF}(M^n) \rightarrow \mathcal{M}_{Fr}^*(M^n, S_1^{n+1})$) is a surjective two-fold map. An admissible GCF-manifold has no singular points if and only if the positive Gauss map $G_+ : M^n \rightarrow S^n$ is an immersion.*

Remark 4.3. An embedding $f : S^n \rightarrow H^{n+1}$ is called *horo-regular* if at least one of the hyperbolic Gauss maps of f is a diffeomorphism. On the other hand, if f lies in the closure of the interior of the osculating horosphere, f is called *horo-convex* (cf. [3]). A horo-regular horoconvex embedding $f : S^n \rightarrow H^{n+1}$ is called *strictly horoconvex*. In [4], it is pointed out that an embedding $f : S^n \rightarrow H^{n+1}$ is strictly horoconvex if and only if both Gauss maps G_+ and G_- are diffeomorphisms. Several characterizations of horoconvexity are given in [3] and [4]. When $f \in \mathcal{M}_{Fr}^*(M^n, H^{n+1})$ is a horo-regular embedding, [4] showed that there is a conformally flat metric g on S^n realizing f . Our map Ψ is a generalization of this procedure in [4]. In [4], horo-regularity (resp. strict horoconvexity) is called horospherical convexity (resp. strongly H -convexity). When f is a front in H^{n+1} , then parallel family of wave front $\{f_\delta\}_{\delta \in \mathbf{R}}$ is induced. Like as in the case of horo-regular hypersurfaces in [4], f_δ induces an admissible GCF-manifold whose metric is a scalar multiple of the metric $\langle \cdot, \cdot \rangle$ of the GCF-manifold induced by f .

Corollary 4.4 (Kuiper [9]). *Let M^n ($n \geq 3$) be a compact 1-connected manifold. Then M^n admits a conformally flat metric if and only if M^n is diffeomorphic to S^n .*

Proof. Suppose that there is a conformally flat metric g on M^n , then $G_+ : M^n \rightarrow S^n$ is an immersion. Since M^n is compact, it gives a finite covering map. Since M^n and S^n are both 1-connected, G_+ must be bijective. \square

Contrary to the above assertion, we can prove the following

Proposition 4.5. *There is a compact 1-connected admissible generalized conformally flat manifold that is not homeomorphic to a sphere.*

Proof. Take a generalized Clifford torus $S^2 \times S^{n-2}$ in S^{n+1} . By a conformal transformation, we can get a hypersurface immersed in an open hemisphere which can be identified with the hyperbolic space H^{n+1} . Then, we get an immersion $f : S^2 \times S^{n-2} \rightarrow H^{n+1}$ with a unit normal vector field ν . Then $x := (f - \nu)/\sqrt{2}$ gives a front in Q_+^{n+1} and the metric $\langle dx, dx \rangle$ induces the desired generalized conformally flat structure on $S^2 \times S^{n-2}$. \square

Corollary 4.6. *There is a spacelike front in Q_+^{n+1} which is not an immersion.*

Proof. Let $x : S^2 \times S^{n-2} \rightarrow Q_+^{n+1}$ be the spacelike front as in the proof of Proposition 4.5. If x is an immersion, then the corresponding compact 1-connected admissible generalized conformally flat manifold has no singularity. Then by Kuiper's theorem (i.e. Corollary 4.4), it is diffeomorphic to S^n , which makes a contradiction. \square

At the end of this section, we discuss conformal changes of a given front in Q_+^{n+1} .

Proof of Proposition 2.7. Since the assertion is a local property, we may assume that M^n is simply connected. Then by Lemma 4.1, there exists a spacelike frontal $x : M^n \rightarrow Q_+^{n+1}$ which induces $(M^n, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$. We denote by $y : M^n \rightarrow Q_-^{n+1}$ its dual. Then

$$\tilde{x} := e^\sigma x : U \longrightarrow Q_+^{n+1}$$

gives a new immersion whose first fundamental form is given by

$$\tilde{g} = \langle d\tilde{x}, d\tilde{x} \rangle = e^{2\sigma} \langle dx, dx \rangle = e^{2\sigma} g.$$

Let $\tilde{\mathcal{E}}$ be the subbundle of $\tilde{x}^* TQ_+^{n+1}$ perpendicular to y , and \tilde{D} be an induced connection on $\tilde{\mathcal{E}}$ with respect to the canonical connection on \mathbf{L}^{n+2} . Then it induces a new GCF-manifold. \square

Let U be a domain in S^n , and $x : U \rightarrow Q_+^{n+1}$ a canonical embedding, that is,

$$x = \begin{pmatrix} 1 \\ p \end{pmatrix} \in \mathbf{L}^{n+2}$$

and $p : U \rightarrow S^n \subset \mathbf{R}^{n+1}$ is the canonical inclusion. Then

$$\tilde{x} := e^\sigma x : U \longrightarrow Q_+^{n+1}$$

gives a new immersion whose first fundamental form is given by

$$\tilde{g} = \langle d\tilde{x}, d\tilde{x} \rangle = e^{2\sigma} \langle dx, dx \rangle = e^{2\sigma} g,$$

where g is the induced metric of U from the unit sphere S^n . Recall that the dual of \tilde{x} is a map $\tilde{y} : U \rightarrow Q_-^{n+1}$ such that

$$\langle \tilde{x}, \tilde{y} \rangle = 1, \quad \langle d\tilde{x}, \tilde{y} \rangle = \langle d\tilde{y}, \tilde{x} \rangle = 0.$$

Then one can directly verify that

$$(4.5) \quad \tilde{y} = -\frac{1}{2}\Delta\tilde{x} - \frac{\langle \Delta\tilde{x}, \Delta\tilde{x} \rangle}{8}\tilde{x}$$

$$(4.6) \quad = \frac{e^{-\sigma}}{2} \left\{ \begin{pmatrix} -1 \\ p \end{pmatrix} - |d\sigma|^2 \begin{pmatrix} 1 \\ p \end{pmatrix} - 2 \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \right\}$$

holds, where

$$\Delta := \sum_{i,j} g^{ij} \nabla_j \nabla_i, \quad |d\sigma|^2 := \sum_{i,j} g^{ij} \sigma_i \sigma_j, \quad \alpha := \sum_{j,k} g^{jk} \sigma_j p_k : U \rightarrow \mathbf{R}^{n+1}.$$

Here, g_{ij} ($i, j = 1, \dots, n$) are the components of the metric g with respect to a local coordinate system (u^1, \dots, u^n) , (g^{ij}) is the inverse matrix of (g_{ij}) , $\sigma_j = \partial\sigma/\partial u^j$ and $p_j = \partial p/\partial u^j$. The first equation (4.5) is the formula in [15].

In particular, the second fundamental form of \tilde{x} is given by

$$II = -\langle d\tilde{x}, d\tilde{y} \rangle = \left(\frac{1 + |d\sigma|^2}{2} \right) g + d\sigma \otimes d\sigma - \text{Hess}(\sigma).$$

The symmetric covariant tensor II satisfies the Codazzi equation, since symmetricity of ∇II is equivalent to the condition (2.3) for the second homomorphism.

Remark 4.7. The frontal bundle $(M^n, \tilde{\mathcal{E}}, e^{2\sigma} \langle \cdot, \cdot \rangle, D, d\tilde{x})$ constructed in the proof of Proposition 2.7 might not be admissible, since \tilde{y} given in (4.6) can diverge if \tilde{x} admits singular points. This generalization of the conformal change of Riemannian metrics is somewhat related to the hyperbolic Christoffel problem posed in [4].

5. ISOMETRIC IMMERSIONS OF A RIEMANNIAN 2-MANIFOLD INTO Q_+^3

If $n \geq 3$, any simply connected conformally flat Riemannian n -manifold is uniquely immersed in the lightcone Q_+^{n+1} . However, if $n = 2$, the situation is different. In this section, we show that there is infinite dimensional freedom for isometric immersions, unless the given simply connected Riemannian 2-manifold is homeomorphic to S^2 . More precisely, we can determine the moduli of the set of immersions of a given simply connected Riemannian manifold into the 3-dimensional lightcone Q_+^3 . First, we prove the following:

Proposition 5.1. *Let (M^2, g) be a 1-connected Riemannian 2-manifold. Then there is an isometric embedding $x : (M^2, g) \rightarrow Q_+^3$.*

Proof. The well-known uniformization theorem implies that (M^2, g) is conformally equivalent to the sphere S^2 , the plane \mathbf{C} or the unit disc \mathcal{D}^2 . Since \mathbf{C} and \mathcal{D}^2 are conformally embedded in the unit sphere S^2 , there is a conformal embedding

$$x_1 : (M^2, g) \longrightarrow S^2.$$

Since x_1 is conformal, there is a smooth function $\sigma \in C^\infty(M^2)$ such that $g = e^{2\sigma} g_1$, where g_1 is a metric of constant Gaussian curvature 1 induced by x_1 . Then we set

$$x := e^\sigma x_1 : (M^2, g) \longrightarrow Q_+^3,$$

which gives the desired isometric embedding. \square

On the other hand, as a corollary of Theorem 3.8, the fundamental theorem of surface theory in the lightcone Q_+^3 is stated as follows:

Proposition 5.2. *Let (M^2, g) be a 1-connected Riemannian manifold and II a symmetric covariant tensor on M^2 satisfying the Codazzi equation (i.e. the covariant derivative ∇II with respect to the Levi-Civita connection is a symmetric tensor). If the Gaussian curvature K of g coincides with the trace of $-II$, then there exists an isometric immersion $x : (M^2, g) \rightarrow Q_+^3$ such that the second fundamental form of x coincides with II . Conversely, any isometric immersions of (M^2, g) to Q_+^3 are given in this manner.*

Proof. Let ∇ be the Levi-Civita connection of g , and $\xi : TM^2 \rightarrow TM^2$ the identity map. We also define a map $\zeta : TM^2 \rightarrow TM^2$ so that

$$g(\xi(X), \zeta(Y)) = -II(X, Y).$$

Then $(M^2, TM^2, \langle \cdot, \cdot \rangle, \nabla, \xi, \zeta)$ is a front bundle. In fact, ζ satisfies (2.3) since II is a Codazzi tensor. It is sufficient to show the integrability condition (3.8) is equivalent to $-K = \text{Trace}_I(II)$. To show this, we take a local orthonormal frame field e_1, e_2 on M^2 such that

$$\lambda_j \delta_{ij} = -II(e_j, e_j) = -\langle \xi(e_i), \zeta(e_j) \rangle = -\langle \xi(e_j), \zeta(e_i) \rangle \quad (j = 1, 2).$$

Then substituting $X = e_1, Y = e_2, v = e_2, w = e_1$, (3.8) reduces to

$$K = -\lambda_1 - \lambda_2 = -\text{Trace}_I II,$$

which proves the assertion. \square

Theorem 5.3. *Let (M^2, g) be a 1-connected Riemannian manifold, and $I_{Q_+^3}(M^2, g)$ the set of congruent classes of isometric immersions of (M^2, g) into Q_+^3 . We fix a complex structure of M^2 which is compatible with respect to the conformal structure. Then $I_{Q_+^3}(M^2, g)$ corresponds bijectively to the set of holomorphic 2-differentials on M^2 (cf. [5]). In particular,*

- (1) *if M^2 is conformally equivalent to the sphere S^2 , then $I_{Q_+^3}(M^2, g)$ consists of a point, namely, the canonical isometric embedding of (M^2, g) into Q_+^3 is rigid,*
- (2) *if M^2 is conformally equivalent to the complex plane \mathbf{C} , then $I_{Q_+^3}(M^2, g)$ corresponds bijectively to the set $\mathcal{O}(\mathbf{C})$ of entire holomorphic functions,*
- (3) *if M^2 is conformally equivalent to the unit disc \mathcal{D}^2 , then $I_{Q_+^3}(M^2, g)$ corresponds bijectively to the set $\mathcal{O}(\mathcal{D}^2)$ of holomorphic functions on the disc \mathcal{D}^2 .*

Proof. As shown in Proposition 5.1, there is an isometric embedding $f : M^2 \rightarrow Q_+^3$. In particular, there is a Codazzi tensor $-II_K$ whose trace is equal to the Gaussian curvature K of the metric g . Then, by Proposition 5.2, $I_{Q_+^3}(M^2, g)$ can be identified with the set $\text{Cod}_{-K}(M^2, g)$ of Codazzi tensors on (M^2, g) whose traces are equal to $-K$. We denote by $\text{Cod}_0(M^2, g)$ the set of traceless Codazzi tensors on (M^2, g) . The following map is a bijection;

$$(5.1) \quad \text{Cod}_0(M^2, g) \ni B \mapsto B + II_K \in \text{Cod}_{-K}(M^2, g).$$

Since M^2 is simply connected, it can be considered as a Riemann surface biholomorphic to S^2, \mathbf{C} or \mathcal{D}^2 . As shown in [16], the set of traceless Codazzi tensors on a compact Riemann surface is identified with the set of holomorphic 2-differentials on it. Thus, if $M^2 = S^2$, the vector space $\text{Cod}_0(M^2, g)$ is zero-dimensional, that is, f is rigid. So we consider the remaining cases $M^2 = \mathbf{C}$ and $M^2 = \mathcal{D}^2$ with the canonical complex coordinate z . We denote by $\mathcal{O}(M^2)$ the set of holomorphic functions on M^2 . Then

$$(5.2) \quad \mathcal{O}(M^2) \ni \varphi(z) \mapsto \varphi(z)dz^2 + \overline{\varphi(z)}d\bar{z}^2 \in \text{Cod}_0(M^2, g)$$

is a linear isomorphism. (The set of holomorphic 2-differentials on M^2 can be identified with $\mathcal{O}(M^2)$, since there is a globally defined holomorphic 2-differential dz^2 on M^2 .) Thus, combining the two maps (5.1) and (5.2), $\mathcal{O}(M^2)$ can be identified with $I_{Q_+^3}(M^2, g)$. \square

Let M^2 be a 2-manifold and $x : M^2 \rightarrow Q_+^3$ an immersion. Then there exists the dual as a C^∞ -map $y : M^2 \rightarrow Q_-^3$ such that $\langle x, y \rangle = 1$ and $\langle dx, y \rangle = \langle x, dy \rangle = 0$.

Even if x is an immersion, the dual y of x may have singular points. In fact, y is a spacelike front in Q_-^3 in general. The first author [13] and Izumiya-Saji [8] pointed out that if x has zero Gaussian curvature with respect to the induced metric, so does y on its regular set. It should be remarked that this duality on flat surfaces corresponds to the following intrinsic duality:

Proposition 5.4. *Let (M^2, g) be a flat Riemannian 2-manifold and II a traceless Codazzi tensor of (M^2, g) . Then the metric \check{g} defined by (2.4) is also flat on the regular set of \check{g} . Moreover, II is also a traceless Codazzi tensor with respect to \check{g} on the regular set of \check{g} .*

Proof. This corresponds to the case of $n = 2$ of our intrinsic duality. (As seen above, the set of traceless Codazzi tensors on (M^2, g) can be identified with $I_{Q_+^3}(M^2, g)$ if M^2 is simply connected.) The proof is parallel to that of Lemma 4.1. \square

Remark 5.5. Let ∇ and $\check{\nabla}$ be the Levi-Civita connections of g and \check{g} respectively. Then $(II, \check{\nabla})$ is the dual Codazzi structure of the Codazzi structure (II, ∇) in the sense of Shima [17].

Several examples of flat surfaces in Q_+^3 are given by the first author [14]. Recently, Izumiya and Saji [8] showed that linear Weingarten fronts in Q_+^3 correspond to linear Weingarten fronts in H^3 as a Legendrian duality. In this setting, one can easily check that flat fronts in Q_+^3 corresponds to flat fronts in H_+^3 . On the other hand, Gálvez, Martínez and Milán [6] found a holomorphic representation formula for flat surfaces in H^3 using a Hessian structure induced by the above Codazzi structure (II, ∇) . Like as in the case of $n \geq 3$, this duality corresponds to the following two-fold map between two sets of flat fronts in Q_+^3 and in H^3 on M^2

$$\mathcal{M}_{Fr}^0(M^2, Q_+^3) \ni (x, y) \mapsto \frac{1}{\sqrt{2}}(x - y, x + y) \in \mathcal{M}_{Fr}^0(M^2, H^3).$$

Several examples and global properties of flat fronts in H^3 are given in [10], [11] and [12].

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